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Knot Symbols: A Tool to Describe and Simplify Knot Diagrams

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A convenient representation of knot diagrams by abstract symbols is introduced. A set of simple moves, which are convenient combinations of the classic Reidemeister moves, is also introduced. These moves can be applied directly to the symbols to obtain simplified symbols (and therefore simplified diagrams) for the knot.

1. INTRODUCTION

In recent years there has been a new wave of interest in knot theory because of significant mathematical progress and unsuspected connections with modern physics (Kauffman, 1991; Lickorish, 1988; Burde and Zieschang, 1986; Birman, 1991; Wadati, 1993). Nevertheless the main problem in knot theory, simply referred to as the *knot problem* (Birman, 1991), is not yet solved. This problem consists in finding a procedure to decide whether two given knots are the same knot (in other words, whether one of them can be continuously deformed into the other). Of course we know that two knot diagrams (i.e., planar projections of knots) are topologically equivalent (they correspond to the same knot) iff one can be obtained from the other through a sequence of Reidemeister moves; however, there is no constructive way to do this or even to establish if this can be done.

Remarkable progress has been made with the introduction of more and more sophisticated invariants of the knot, namely quantities that do not change while deforming the original diagram through the Reidemeister moves (Kauffman, 1991; Lickorish, 1988; Burde and Zieschang, 1986; Birman, 1991); if some invariant is not the same for two knot diagrams, one can say

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that the two diagrams do not correspond to the same physical knot. On the other hand, if it happens (and it can happen also for simple knots) that all the known invariants are the same for the two diagrams, nothing can be said. Thus, up to now, knot invariants are not the definitive answer to the knot problem (we do not have a complete set of invariants). Moreover, the more powerful they are (consider, for instance, the HOMFLY Polynomials), the more cumbersome their construction becomes.

In this paper we propose an alternative route, introducing *knot symbols*, a simple but effective tool to treat knot diagrams. In short, we associate with a given knot diagram a knot symbol, namely a string of marked letters. Then we can apply the Reidemeister moves, or better, new, simple moves that are convenient combinations of Reidemeister moves, *directly* and in a very simple way to the knot symbols themselves, obtaining new knot symbols which are equivalent to the previous ones (the corresponding diagrams being topologically equivalent).

Moreover, an effective procedure is introduced to obtain from a given knot symbol (knot diagram) a new, equivalent one with a lesser (or at least not greater) number of letters (crossings). We do not know if the final reduced knot symbol that we get when the procedure stops is minimal, i.e., if the corresponding reduced diagram is indeed really not further reducible. Nevertheless, we stress the obvious advantages of obtaining a reduced diagram using a procedure which is very fast and can be used very easily by hand and in addition can be exploited by computer. Moreover, within bounds specified in the following, knot symbols also seem to be a promising tool to detect chirality and to distinguish diagrams which have the same HOMFLY Polynomial.

Knot symbols can be considered as a generalization of the notation first introduced by Dowker and Thistlethwaite (1983; Thistlethwaite, 1985) (this notation was unknown to the author when the first version of this paper was written). However, the author thinks that is still worthwhile to present his knot symbols to an audience of theoretical physicists for the following reasons:

- The generalization introduced allows one to handle the problem of *chirality*, and this could be a considerable advantage in physical applications of the knot theory.
- The introduced *reduction procedure* is simple and effective, and it could be profitably used in applications, possibly via computer implementation.
- In this paper a number of problems and difficulties of knot theory which are well known to specialists but possibly not to physicists are considered and reviewed in the knot symbol notation.

In Section 2 we show how to construct the knot symbol corresponding to a given knot diagram, then we give some elementary equivalence rules between knot symbols and we show how to reconstruct a diagram from a knot symbol; in Section 3 we give four simple moves to modify knot symbols into equivalent ones; in Section 4 we introduce our procedure to simplify knot symbols; in Section 5 we give examples; Section 6 is devoted to final remarks.

2. KNOT SYMBOLS

Given an oriented diagram for a knot (Kauffman, 1991) (we use as an example diagram F1 in Fig. 1), construct a knot symbol corresponding to such a diagram following the following procedure:

R1. Start from any point of the diagram and reach, following the given orientation, the first crossing (in diagram F1 start from O).

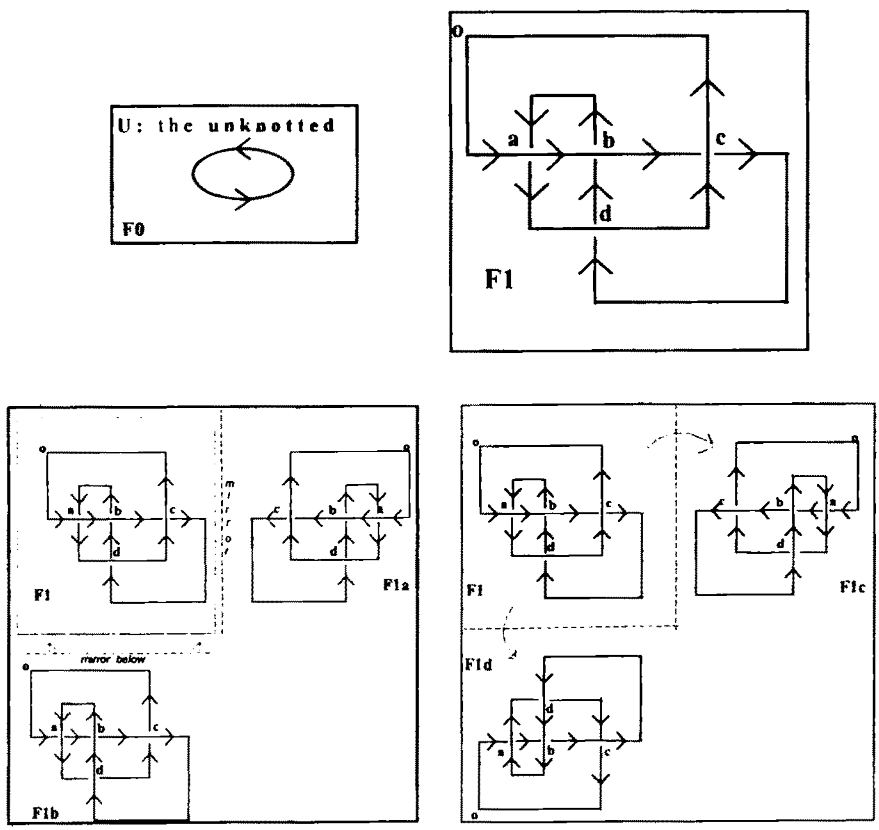

Fig. 1.

R2. Assign a letter to the crossing (as a label or "name" for the crossing: in F1, "a" for the first crossing).

R3. Add this letter to the string which will constitute the knot symbol and (**a**) underline the letter if you are crossing underneath (in F1 you are crossing over), and (**b**) put a tilde over the letter if you see the other incoming arrow on your left (in F1 you have to put a tilde on the label for the first crossing, $\tilde{a}$).

R4. Go on to the next crossing and repeat rules R2 and R3a,b as necessary until you reach again the starting point (for the knot in diagram F1 this gives the knot symbol S1 = $\tilde{a}b\underline{c}\underline{\tilde{d}}\underline{\tilde{b}}\underline{a}d\tilde{c}$).

Note that following the above four rules, no symbol can be constructed for the unknotted case, the knot with no crossing (see diagram F0). It is convenient to put this as Rule 0:

R0. The knot symbol corresponding to the "unknotted" case is the null knot symbol (an empty string): S(U) = ∅.

It is also convenient to introduce the following definition:

D0. In a knot symbol we call a *label* any letter plus its status with regard to underlining and having a tilde; moreover, we call *homologous labels* two labels in a knot symbol with the same letter.

Example. In S1, $\tilde{a}$ and $\underline{c}$ are different labels; $\tilde{a}$ and $\underline{a}$ are homologous labels.

Remarks. (i) In a knot symbol each letter obviously appears twice; moreover, if the first time it is underlined, the second time it is not (and vice versa), and if the first time it has a tilde, the second time it has not (and vice versa).

(ii) Many alternative, possibly simpler, notations could be easily devised. For computer applications and for extensions to links, a more convenient notation seems to be the following: each label is given by the same letter with two indices, the first giving by its value the number of the crossing and by its sign the status with regard to underlining, the second giving the status with regard to whether or not it has a tilde (0,1). Using this notation, the knot symbol S1 reads

$$A(+1,1)A(+2,0)A(-3,0)A(-4,1)A(-2,1)A(-1,0)A(+4,0)A(+3,1)$$

(iii) Note the analogies with Kauffman labeling introducing bracket polynomials (Kauffman, 1991).

2.1. Elementary Equivalence Rules and Definitions

A knot symbol should be considered as a representative of a class of equivalent symbols that correspond possibly to different diagrams but to the same (topological) knot; we give now three elementary rules of equivalence (the equivalence between knot symbols will be denoted by $\approx$):

ER1. Upon replacing a letter in a knot symbol with a different one (not already present in the symbol!) or exchanging two letters, we obtain an equivalent knot symbol.

Example.

$$\mathrm{S1:}\quad \tilde{a}b\underline{c\tilde{\tilde{d}}\tilde{\tilde{b}}a}d\tilde{c} \approx \tilde{a}f\underline{c\tilde{\tilde{d}}\tilde{\tilde{f}}a}d\tilde{c} \approx \tilde{\tilde{f}}a\underline{c\tilde{\tilde{z}}\tilde{a}f}z\tilde{c}$$

ER2. Knot symbols are cyclic: knot symbols obtained from one another by permuting the labels in the string are equivalent.

Remark. This obviously takes into account the arbitrariness of the choice of the starting point.

Example.

$$\mathrm{S1:}\quad \tilde{a}b\underline{c\tilde{\tilde{d}}\tilde{\tilde{b}}a}d\tilde{c} \approx \underline{c\tilde{\tilde{d}}\tilde{\tilde{b}}a}d\tilde{c}\tilde{a}b \approx \underline{a}d\tilde{c}\tilde{a}b\underline{c\tilde{\tilde{d}}\tilde{\tilde{b}}}$$

ER3. Two knot symbols obtained from one another by reversing the order of the labels are equivalent.

Remark. Of course this corresponds to a change in the orientation of the knot diagram; this rule should not be considered if one needs to distinguish diagrams differing only in the orientation.

Example.

$$\mathrm{S1:}\quad \tilde{a}b\underline{c\tilde{\tilde{d}}\tilde{\tilde{b}}a}d\tilde{c} \approx \tilde{c}d\underline{a\tilde{\tilde{b}}\tilde{\tilde{d}}c}b\tilde{a}$$

Now let us introduce some definitions we will use in the following:

D1. Given a knot symbol, say S, we define its *up conjugate* $\tilde{S}$ as the knot symbol obtained from S by inverting the operation of giving and removing tildes.

Example.

$$\mathrm{S:}\quad \tilde{a}b\underline{c\tilde{\tilde{d}}\tilde{\tilde{b}}a}d\tilde{c} \Rightarrow \tilde{\mathrm{S}}\mathrm{:}\quad a\tilde{b}\underline{\tilde{c}db\tilde{a}}\tilde{d}c$$

D2. Given a knot symbol, say S, we define its *down conjugate* $\underline{\mathrm{S}}$ as the knot symbol obtained from S by inverting the "underlining" procedure.

Example.

$$S: \quad \tilde{a}b\underline{c}\underline{\tilde{d}}\underline{\tilde{b}}ad\tilde{c} \Rightarrow \underline{S}: \quad \underline{\tilde{a}}\underline{b}c\tilde{d}\tilde{b}a\underline{d}\underline{\tilde{c}}$$

D3. Given a knot symbol, say S, we define its *full conjugate* $\underline{\tilde{S}}$ as the knot symbol obtained from S by inverting the "underlining" and the "tilde" operations.

Example.

$$S: \quad \tilde{a}b\underline{c}\underline{\tilde{d}}\underline{\tilde{b}}ad\tilde{c} \Rightarrow \underline{\tilde{S}}: \quad \underline{a}\underline{\tilde{b}}\tilde{c}db\tilde{a}\underline{\tilde{d}}\underline{c}$$

Note that the up and the down conjugate knot symbols correspond to mirror images of the knot: indeed it is easy to verify that the above up conjugate knot symbol $\tilde{S}$ corresponds to diagram F1a of Fig. 1, which is obtained from diagram F1 of Fig. 1 through an inversion along a vertical axis and it is just the image of diagram F1 as seen in a mirror orthogonal to the plane of the diagram itself, while the down conjugate $\underline{S}$ corresponds to diagram F1b of Fig. 1, which is obtained by switching all the crossings of diagram F1 and it is the diagram of the *physical* knot F1 as viewed on a mirror placed below the knot itself. The full conjugate $\underline{\tilde{S}}$ corresponds to the diagram of the *physical* knot F1 turned over on the plane after a π rotation in the space (diagrams F1c and F1d of Fig. 1).

Summarizing, a *partially conjugate* knot symbol corresponds to a mirror image of the knot, while the *full conjugate* knot symbol corresponds to the same knot; thus:

ER4. A knot symbol S and its full conjugate $\underline{\tilde{S}}$ are equivalent: $S \approx \underline{\tilde{S}}$.

We need some more definitions:

D4. Two contiguous labels in a knot symbol form a *down* (*up*) *permanence* $\underline{P}$ ($\tilde{P}$) iff they have the same state of underlining (tilding). Moreover, a permanence is *negative* (P−) if the two letters are not underlined (tilded), *positive* (P+) otherwise.

Example. In the knot symbol S1 the letters (a,c) form a $\underline{P}-$; (c,d) form a $\underline{P}+$ and also a $\underline{P}-$; (d,b) form a $\tilde{P}+$.

D5. Two contiguous labels which do not form a down (up) permanence in a knot symbol form a *down* (*up*) *alternating*; moreover, a knot symbol itself without down (up) permanencies is said to be *down* (*up*) *alternating*.

Remark. Indeed a down alternating knot symbol corresponds to an alternating knot (Kauffman, 1991).

Example.

$$\text{S2:}\quad \underline{a}\tilde{b}\underline{c}\tilde{a}\underline{b}\tilde{c}$$

(this is a down and up alternating knot symbol that corresponds to the so-called *trefoil knot*; see below).

D6. In a knot symbol two contiguous homologous labels form a *noose.*

Remark. Of course a noose is also an up and down alternating.

Examples: $\ldots \tilde{x}\underline{x} \ldots; \ldots \underline{\tilde{y}}y \ldots; \ldots \underline{z}\tilde{z} \ldots.$

D7. In a knot symbol we call a *snare* the configuration of two couples of homologous labels that form two down permanencies (of course a positive and a negative one) and two up alternatings.

Examples: $\ldots x\tilde{y} \ldots \underline{\tilde{x}y} \ldots; \ldots \tilde{a}b \ldots \underline{\tilde{b}a} \ldots; \ldots \underline{f\tilde{g}} \ldots g\tilde{f} \ldots.$

D8a. In a knot symbol we call a *quasi-chain* a configuration of three couples of homologous labels that form two down permanencies (of course a positive and a negative one) and one down alternating (not a noose). The two down permanencies and the down alternating are called *links* of the quasi-chain.

Examples:

$$c1 = \ldots \underline{\tilde{x}\tilde{y}} \ldots x\underline{\tilde{z}} \ldots zy \ldots$$

$$c2 = \ldots \underline{x}\tilde{y} \ldots \underline{zy} \ldots \tilde{x}\tilde{z} \ldots$$

$$c3 = \ldots \tilde{z}y \ldots \underline{z}x \ldots \underline{\tilde{x}\tilde{y}} \ldots$$

$$c4 = \ldots x\underline{\tilde{z}} \ldots \underline{\tilde{x}y} \ldots \tilde{y}z$$

D8b. In a quasi-chain we define as the *chain parity* of a label the number $n = \{(p + t + a) \bmod 2\}$, where $p = 0$ ($p = 1$) if the label is the first (second) in a "link," $t = 1$ ($t = 0$) if the letter in the label is (is not) tilded, and $a = 1$ ($a = 0$) if the label is in a "link" that is (is not) an up alternating.

Examples. Consider the label $\underline{\tilde{z}}$ in the above quasi-chain c1: for this label we have $p = 1, t = 1, a = 1$, so that the chain parity is $n = (1 + 1 + 1) \bmod 2 = 1$; the homologous label has chain parity $n = (0 + 0 + 0) \bmod 2 = 0$.

D8c. In a knot symbol we call a *chain* a quasi-chain whose homologous labels have the same chain parity.

Examples. It is easy to check that considering the previous examples of quasi-chains, only c2 and c3 are chains.

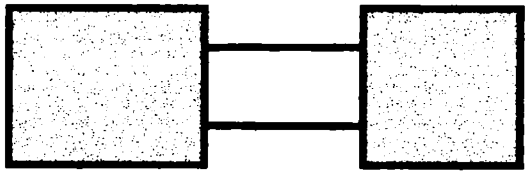

Fig. 2.

Remark. According to D8c, only the status of tilding distinguishes between a chain and a quasi-chain: it is straightforward (if tedious) to check that, up to the (here unimportant) ordering of the "links" and taking into account the previous equivalence rules, all the tilding configurations admissible for a chain reduce to the following two:

$$\ldots \tilde{x}y \ldots xz \ldots \tilde{z}\tilde{y} \ldots; \quad \ldots \tilde{x}y \ldots \tilde{z}x \ldots \tilde{y}z \ldots$$

In other words, a quasi-chain is indeed a chain iff applying the equivalence rules it is possible to recover one of the above two tilding configurations (up to the order of the "links").

D9. We call a *segment* a substring s of contiguous labels in the string which constitutes the knot symbol S (s ⊂ S); a segment containing only couples of homologous labels is called an *isle*.

Remarks. (a) Of course S itself is an isle; a (nonempty) isle s different from S is called a *proper isle*.

(b) If the knot symbol S contains a proper isle s, then it contains also its *complementary isle*, that is, the proper isle $s' = S - s$.

(c) A diagram with a proper isle can be block-cast as in Fig. 2.

(d) An isle may contain *sub-isles*.

Examples. (a) Consider again S1:

s1: $bc\underline{\tilde{d}\bar{\tilde{b}}}$ is a segment of S1

s2: $d\bar{c}\tilde{a}$ is a segment of S1 (consider ER2)

s3: $\underline{c\tilde{b}a}$ is not a segment of S1 (lack of the contiguity requirement)

(b) s1, s2, s3 are not isles.

(c) The knot symbol for the diagram in Fig. 3 contains an isle.

2.2. The Inverse Problem

Let us now briefly consider the "inverse problem," i.e., how to reconstruct a knot diagram from its knot symbol. Indeed this task can be easily accomplished when the symbol is truly a knot symbol, namely if it really

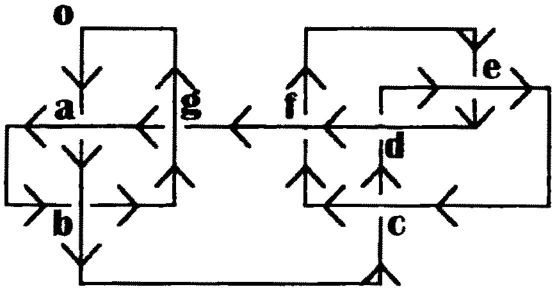

Fig. 3. The knot symbol for this diagram is S3: $\underline{\tilde{a}}b\underline{c}\underline{d}\tilde{e}\tilde{c}\underline{f}\underline{e}\tilde{d}\tilde{f}\tilde{g}a\underline{\tilde{b}}g$; with s: $\underline{c}\underline{d}\tilde{e}\tilde{c}\underline{f}\underline{e}\tilde{d}\tilde{f}$ and s′: $\underline{\tilde{g}}a\underline{\tilde{b}}g\underline{\tilde{a}}b$. Here s is a proper isle and s′ is its complementary isle.

comes from a knot diagram; e.g., let us consider the above knot symbol S2 = $\underline{a}\tilde{b}\underline{c}\tilde{a}\underline{b}\tilde{c}$ for the trefoil knot and let us try to construct from it the diagram for the trefoil knot itself.

Let us start from the first letter (a) and try to construct the first crossing: we know that this is an under crossing (the first a is underlined); moreover, we must see the arrow in the other over crossing line on our right (a is not tilded); so we get the diagram in Fig. 4A. We do the same for the next two letters b and c (see Fig. 4B); then we have to reach again the letter a entering the crossing in the direction indicated by the arrow (see Fig. 4C). Now we complete the diagram reaching b, c, and finally a (see Fig. 4D), obtaining the diagram of the *trefoil knot* or *simple knot to the left* (see also the following). Note that we have a different path to go from $\underline{c}$ to $\tilde{a}$ yielding the diagram in Fig. 5: this different (but topologically equivalent) diagram has the same knot symbol and corresponds obviously to the same knot.

One might wonder if, given an arbitrary string of letters (each letter repeated twice and underlined and tilded only once), it is always possible to

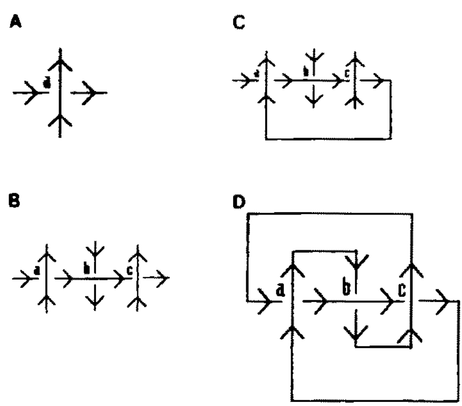

Fig. 4.

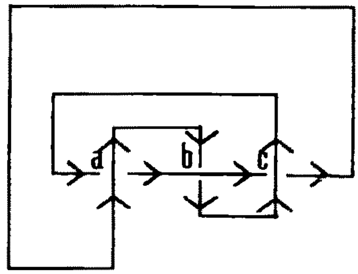

Fig. 5.

reconstruct a knot. The answer is obviously "no"; it is sufficient to give a counterexample.

Consider the knot symbol S: $\underline{a}\underline{b}\tilde{c}\underline{\tilde{a}}\underline{c}\tilde{b}$; following the previously outlined procedure, one can easily reconstruct the knot diagram up to $\tilde{b}$ (see Fig. 6). Now it is evidently impossible to reach $\underline{a}$ without creating a new, unwanted crossing. Note that this impossibility is not due to the choice in going, for instance, from $\tilde{c}$ to $\tilde{a}$, or from $\tilde{a}$ to $\underline{c}$ (try!), and, moreover, it is not due to the given orientation (structure of "tilding"): indeed it is easy to see that a *necessary* (but not *sufficient*) condition for a string to be actually a knot symbol is that between two homologous labels there must be an even number ($2r$) of labels ($r = 0, 1, 2, \ldots$). This property is violated in the above example. The problem of what structure of a knot symbol is admissible is an open problem [this problem is solved for Dowker's notation: indeed in this case it reduces to the well-known *crossing sequence problem* (Dowker and Thistlethwaite, 1983)].

3. MOVES IN KNOT SYMBOLS

We will give a set of four *moves*, i.e., operations that one can perform to modify a knot symbol into a new one which corresponds to a different knot diagram but to the same topological knot: thus we will introduce nontrivial

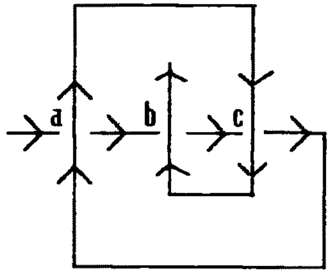

Fig. 6.

equivalences between knot symbols. We will illustrate the moves with appropriate diagrams and we will show also how and to what extent our moves include the celebrated Reidemeister moves.

Move α1. Any isle can be moved in any position in the knot symbol

Remark. This corresponds to a slipping of the isle in the knot.

Example. Consider the diagram in Fig. 3 with its proper isle and its knot symbol S3: $\underline{\tilde{a}}b\mathrm{I}\underline{\tilde{g}}a\underline{\tilde{b}}g$ (here I stands for the proper isle I: $\underline{cd}\tilde{e}\tilde{c}\underline{fe}\tilde{d}\tilde{f}$; see Fig. 7).

Using move α1, one can obtain, for instance, S′3: $\underline{\tilde{a}}b\tilde{g}a\underline{\tilde{b}}g\mathrm{I}$ (S′3 ≈ S3), which corresponds to the new diagram in Fig. 8: this can be done physically by slipping the isle through, e.g., $b\underline{\tilde{a}}$.

Move α2. Any isle can be fully conjugate, independent of the remainder of the knot symbol.

Remark. This corresponds to a π rotation in the space of the isle itself (see Fig. 9).

Example. Consider S′3; α2 yields the diagram in Fig. 10: the corresponding knot symbol is S″3: $\underline{\tilde{a}}b\underline{\tilde{g}}a\underline{\tilde{b}}g\tilde{c}\tilde{d}\underline{ec}\tilde{f}\tilde{e}\underline{df}$ (S″3 ≈ S′3 ≈ S3).

Remark. One can move an isle I_1 inside another isle I_2: if I_2 is different from the complementary isle of I_1, then I_2 is destroyed as an isle, or better, a new, bigger isle, say I_2', is created with a sub-isle I_1.

Move β. The labels of the three "links" of a chain can be commuted.

Remark. Of course the new quasi-chain is indeed still a chain, because by commuting the labels of all the "links" we just invert the chain parity of all the labels.

Examples:

[i] $\ldots \tilde{x}y \ldots \underline{\tilde{z}\tilde{y}} \ldots \underline{x}z \ldots \Rightarrow \ldots y\tilde{x} \ldots \underline{\tilde{y}\tilde{z}} \ldots z\underline{x} \ldots$

[ii] $\ldots \tilde{x}\underline{\tilde{y}} \ldots yz \ldots \underline{x\tilde{z}} \ldots \Rightarrow \ldots \underline{\tilde{y}}\tilde{x} \ldots zy \ldots \underline{\tilde{z}x} \ldots$

[iii] $\ldots \underline{\tilde{z}y} \ldots \tilde{y}\underline{x} \ldots \tilde{x}z \ldots \Rightarrow \underline{y\tilde{z}} \ldots \underline{x}\tilde{y} \ldots z\tilde{x} \ldots$

This move corresponds to move III of Reidemeister (see Fig. 11A). In terms of knot symbols we have for the move in Fig. 11A

$$\ldots \tilde{x}y \ldots \underline{\tilde{z}\tilde{y}} \ldots \underline{x}z \ldots \Rightarrow \ldots x'\tilde{y}' \ldots \underline{\tilde{x}}'\underline{\tilde{z}} \ldots z\underline{y}' \ldots$$

Now, with $x' = y$ and $y' = x$ (ER1), the right-hand side of [i] is recovered. Note that the resulting knot symbol is the same if we perform the "physical,"

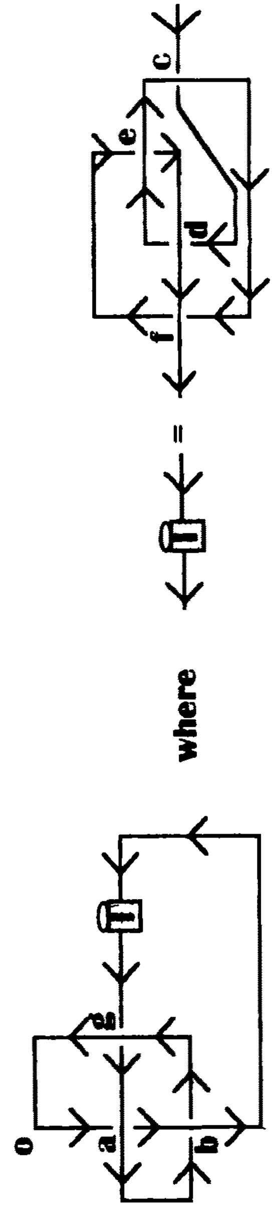

Fig. 7.

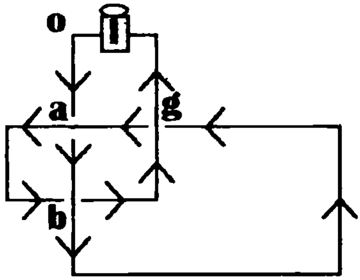

Fig. 8.

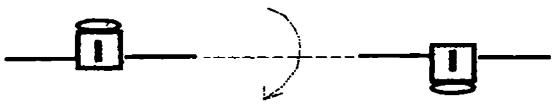

Fig. 9.

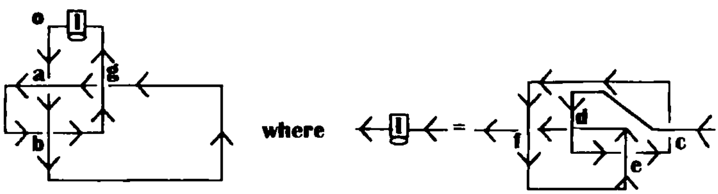

Fig. 10.

"real" move in the other two possible ways. Indeed, following the move in Fig. 11B, in terms of knot symbols we have

$$\ldots \tilde{x}y \ldots \underline{\tilde{z}\tilde{y}} \ldots \underline{x}z \ldots \Rightarrow \ldots z'\tilde{x} \ldots \underline{\tilde{z}}'\underline{\tilde{y}}' \ldots y'\underline{x} \ldots$$

Setting z′ = y and y′ = z again, we have the r.h.s. of [i]. Finally, exploring the last possibility in Fig. 11C, in terms of knot symbols we have

$$\ldots \tilde{x}y \ldots \underline{\tilde{z}\tilde{y}} \ldots \underline{x}z \ldots \Rightarrow \ldots y\tilde{z}' \ldots \underline{\tilde{y}\tilde{x}}' \ldots x'\underline{z}' \ldots$$

Setting z′ = x and x′ = z, the r.h.s. of [i] is obtained.

Now we can add a new (not elementary) equivalence rule between knot symbols:

ER5. Two knot symbols obtained from one another through moves α or β are equivalent.

The moves α and β do not change the number of letters in the knot symbol (i.e., the number of crossings in the knot diagram) even if the new, *equivalent*, knot symbol obtained via the moves looks quite different. On the

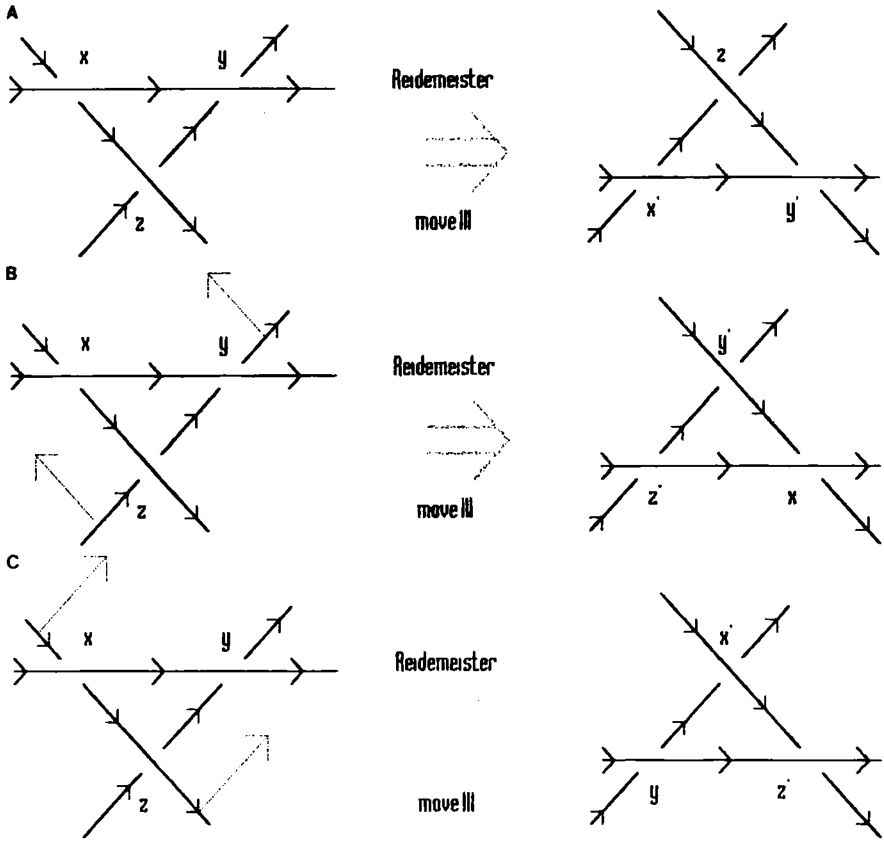

Fig. 11.

contrary, the following two moves simplify the knot symbol decreasing the number of letters (crossings).

Move γ. In a knot symbol a noose can be eliminated.

Examples:

$$\ldots a\underline{b}\tilde{z}x\underline{\tilde{x}}f\tilde{g} \ldots \Rightarrow \ldots a\underline{b}\tilde{z}f\tilde{g} \ldots$$

$$\ldots \tilde{f}\underline{ry}\tilde{y}z\underline{\tilde{h}} \ldots \Rightarrow \ldots \tilde{f}\underline{r}z\underline{\tilde{h}} \ldots$$

This move corresponds to Reidemeister move I (see Fig. 12):

$$\ldots \tilde{z}x\underline{\tilde{x}}f \ldots \Rightarrow \ldots \tilde{z}f \ldots$$

Note, however, that since we are interested in simplifying knot symbols, we do not consider the inverse of this move, even if this could be done: indeed,

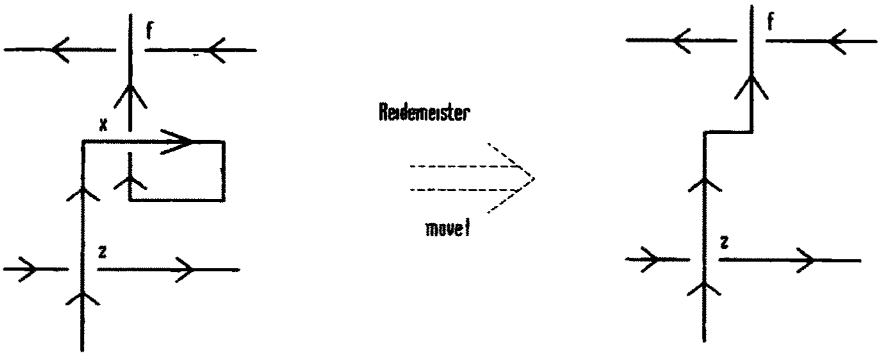

Fig. 12.

obviously it is possible to insert a noose in an arbitrary position of a knot symbol.

Now the fourth and last move:

Move δ. In a knot symbol a snare can be eliminated.

Example (see Fig. 13):

$$\dots \underline{\tilde{f}}\tilde{x}y\underline{\tilde{g}} \dots \underline{\tilde{s}\tilde{y}x}\tilde{r} \dots \Rightarrow \dots \underline{\tilde{f}\tilde{g}} \dots \underline{\tilde{s}}\tilde{r} \dots$$

Note that in the above example we have just used Reidemeister move II; however, our move δ takes into account also the following "physical" move (see Fig. 14):

$$\dots \tilde{f}\underline{x\tilde{y}}\tilde{g} \dots r\tilde{x}y\tilde{s} \dots \Rightarrow \dots \tilde{f}\tilde{g} \dots r\tilde{s} \dots$$

Note also that the other possible way of performing the "physical" move gives the same result. As for our move γ and for the same motivations, also in this case we do not consider the inverse of the move δ: however, it is

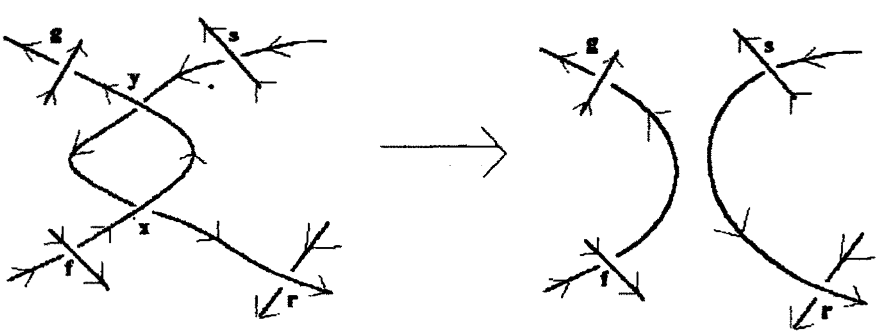

Fig. 13.

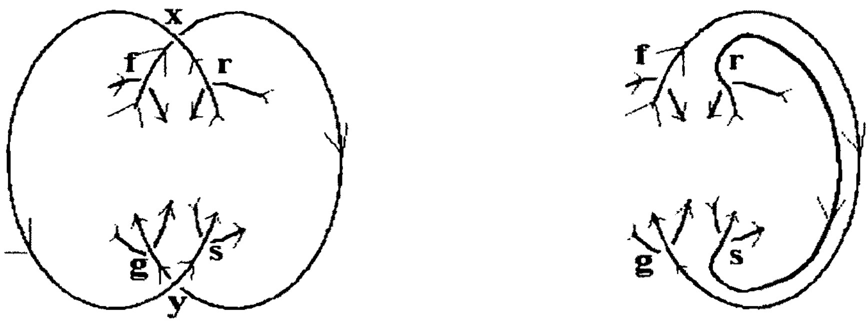

Fig. 14.

important to notice that in this case it is hard to formulate the inverse of move δ in terms of knot symbols (indeed it is easy to check that one cannot simply insert arbitrarily a snare in a given knot symbol; see also considerations at the end of Section 2).

Let us now introduce the last (nontrivial) equivalence rule:

ER6. Two knot symbols obtained from one another through the moves γ and δ are equivalent.

4. A PROCEDURE TO SIMPLIFY KNOT SYMBOLS

4.1. The Procedure

An effective, simple procedure to simplify knot symbols is as follows:

A. Use the moves γ and/or δ (if possible) to simplify.
B. When the moves γ, δ are no longer applicable, identify all the isles (and sub-isles) and use the move α to separate them.
C. Inside an isle repeat the steps A, B if possible; otherwise go to D.
D. Use the move β until you can use again A, B, C: if no possible move β allows you to use again A, B, C, then the isle is called locally irreducible. Go to an unprocessed isle, if any, and there start the procedure again.

Obviously when the procedure stops, we have a knot symbol equivalent to the starting one but with a smaller (or at least not greater) number of letters; from this simplified knot symbol one can reconstruct a diagram for the knot with a smaller (not greater) number of crossings. Of course the final knot symbol is *locally irreducible*, i.e., irreducible with respect to the outlined procedure, and one can hope that it is also minimal. By the way it could happen that starting from the diagram corresponding to what we call a locally irreducible knot symbol for an isle or a knot with, say, N letters (crossings) and using the Reidemeister moves I and II to *add* crossings, the new diagram (symbol) could allow an alternative route to simplify it yielding a final diagram (symbol) with $N' < N$ crossings. We will give in the following a classical example in which this phenomenon happens using the original set of Reidemeister moves (not with our moves!): we were not able to find an analogous example for our moves. We do not dare to hope that our procedure is powerful enough so that what we call locally irreducible is indeed irreducible, but let us hope at least that the first failure that will be found will correspond to an irreducible diagram with a high number of crossings (this could be enough for practical purposes).

Remarks. (i) It should be clear from the above considerations that two equivalent (in terms of our equivalence rules) knot symbols surely correspond to the same topological knot; on the other hand, whenever we speak of different, not equivalent knot symbols, they are not equivalent with respect to our procedure, but we are not sure that these knot symbols correspond indeed to two different knots.

(ii) It is obvious that for a down alternating knot symbol moves β and δ cannot apply (these moves require the presence of down permanencies): thus, if also moves α and γ cannot apply, the down alternating is locally irreducible.

(iii) It is worthwhile to notice that a locally irreducible down and up alternating knot symbol is not equivalent to its up or down conjugate (mirror image).

4.2. Examples

Now we will give examples of the procedure at work.

Let us consider the diagram in Fig. 15 and the corresponding knot symbol

$$\underline{\tilde{a}bc}\tilde{d}\underline{e}f\underline{\tilde{g}}\tilde{e}r\underline{\tilde{s}t}g\underline{\tilde{f}\tilde{r}d}\tilde{x}\tilde{b}y\underline{\tilde{z}}a\underline{x}\tilde{c}s\tilde{t}\underline{\tilde{y}}z$$

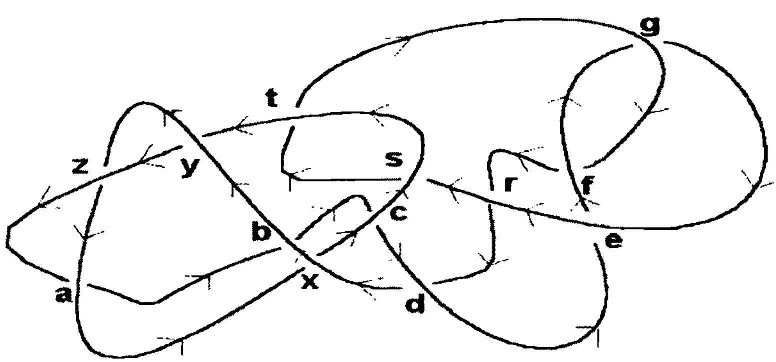

Fig. 15.

We have 13 letters (crossings), and no proper isle. Let us start with step A of our procedure:

- By inspection, no noose is found, so the move γ is not applicable.
- To apply δ we have to look for (double) down permanencies: it is easy to identify the snare . . . $\underline{\tilde{s}t}$ . . . $s\tilde{t}$ . . . , so we cancel the letters s and t using δ. The knot symbol becomes

$$\underline{\tilde{a}bc}\tilde{d}\underline{e}f\underline{\tilde{g}}\tilde{e}rg\underline{\tilde{f}\tilde{r}d}\tilde{x}\tilde{b}y\underline{\tilde{z}}a\underline{x}\tilde{c}\underline{\tilde{y}}z$$

- There is no noose or snare in the above knot symbol, so we go to step B.
- It is possible to identify the isles I_1 and I_2 containing the sub-isle I_3:

$$I_1 = \tilde{x}\tilde{b}y\underline{\tilde{z}}a\underline{x}\tilde{c}\underline{\tilde{y}}z\underline{\tilde{a}bc}; \qquad I_2 = \tilde{d}\underline{e}f\underline{\tilde{g}}\tilde{e}rg\underline{\tilde{f}\tilde{r}d} \supset I_3 = \underline{e}f\underline{\tilde{g}}\tilde{e}rg\underline{\tilde{f}\tilde{r}}$$

- Thus we apply α, obtaining $I_1\tilde{d}\underline{d}I_3$.
- We are now at step C. We consider first the simple central isle and we eliminate the letter d (and the isle as well!) using γ; we have I_1I_3.
- We consider I_1: A, B give no change, so we try to apply the move β (step D). We find the chain $\tilde{x}\tilde{b}$ . . . $\underline{x}\tilde{c}$ . . . $\underline{bc}$; through move β we obtain $I_1 = \tilde{b}\tilde{x}y\underline{\tilde{z}}a\tilde{c}\underline{x\tilde{y}}z\underline{\tilde{a}cb}$.
- Now we start again with step A and we use move γ to eliminate the letter b and the move δ to eliminate the letters x, y and then a, c, getting $I_1 = \underline{\tilde{z}}z$; through the move γ we eliminate also z. Thus the knot symbol for this isle reduces to the null symbol (the unknotted).
- Now we have to consider I_3: A, B give no change, so we go to step D. A chain, namely $f\underline{\tilde{g}}$ . . . rg . . . $\underline{\tilde{f}\tilde{r}}$, is individuated; thus, applying the move β, we get $I_3 = \underline{e\tilde{g}}\tilde{f}\tilde{e}gr\underline{\tilde{r}\tilde{f}}$; [step A] using γ to cancel the letter r and β to cancel the letters e, g and finally using again γ to cancel f, we end with a null knot symbol also for this isle: thus we have established that the original knot is indeed unknotted.

We want to stress that, although for the sake of clarity here we were rather prolix, the procedure is very fast; however, it may be worthwhile to give another example introducing a more concise notation.

We will use different brackets to delimit patterns of interest, namely $[\cdot]$ for isles; $(\cdot)$ for chains; $\{\cdot\}$ for nooses; $\langle\cdot\rangle$ for snares.

Now let us consider the knot diagram in Fig. 16 with 16 crossings and the corresponding knot symbol

$$\underline{\tilde{a}}b\underline{\tilde{z}}c\underline{\tilde{d}}\tilde{e}f\underline{\tilde{g}eh}s\tilde{r}\underline{\tilde{c}\tilde{f}}gd\tilde{h}\tilde{u}vz\underline{r}\tilde{t}\underline{t\tilde{s}\tilde{b}\tilde{v}u}x\underline{\tilde{y}}a\underline{\tilde{x}}y$$

Let us apply the procedure:

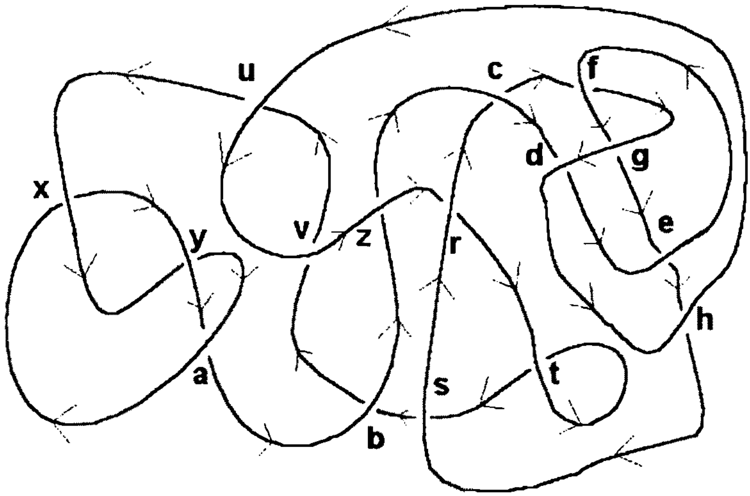

Fig. 16.

Step A:

$$\underline{\tilde{a}}b\underline{\tilde{z}}c\underline{\tilde{d}}\tilde{e}f\underline{\tilde{g}eh}s\tilde{r}\underline{\tilde{c}\tilde{f}}gd\tilde{h}\langle\tilde{u}v\rangle z\underline{r}\{\tilde{t}\underline{t}\}\underline{\tilde{s}\tilde{b}}\langle\tilde{v}u\rangle x\underline{\tilde{y}}a\underline{\tilde{x}}y$$

$$\xrightarrow{\text{moves } \delta \text{ and } \gamma} \underline{\tilde{a}}b\underline{\tilde{z}}c\underline{\tilde{d}}\tilde{e}f\underline{\tilde{g}eh}\langle s\tilde{r}\rangle\underline{\tilde{c}\tilde{f}}gd\tilde{h}z\langle\underline{r\tilde{s}}\rangle\underline{\tilde{b}}x\underline{\tilde{y}}a\underline{\tilde{x}}y$$

$$\xrightarrow{\text{move } \gamma} \underline{\tilde{a}}b\underline{\tilde{z}}c\underline{\tilde{d}}\tilde{e}f\underline{\tilde{g}eh\tilde{c}\tilde{f}}gd\tilde{h}z\underline{\tilde{b}}x\underline{\tilde{y}}a\underline{\tilde{x}}y$$

Step B:

$$b\underline{\tilde{z}}[c\underline{\tilde{d}}\tilde{e}f\underline{\tilde{g}eh\tilde{c}\tilde{f}}gd\tilde{h}]z\underline{\tilde{b}}[x\underline{\tilde{y}}a\underline{\tilde{x}}y\underline{\tilde{a}}]$$

$$\xrightarrow{\text{move } \alpha} [b\underline{\tilde{z}}z\underline{\tilde{b}}][c\underline{\tilde{d}}\tilde{e}f\underline{\tilde{g}eh\tilde{c}\tilde{f}}gd\tilde{h}][x\underline{\tilde{y}}a\underline{\tilde{x}}y\underline{\tilde{a}}]$$

Step C. *First isle.*
Step A:

$$b\{\underline{\tilde{z}}z\}\underline{\tilde{b}} \xrightarrow{\text{move } \delta} \{b\underline{\tilde{b}}\} \xrightarrow{\text{move } \delta} \varnothing$$

Second isle. Steps A, B: no move is possible, thus (step C) we have to go to:

Step D:

$$c(\underline{\tilde{d}}\tilde{e})f(\underline{\tilde{g}e})h\underline{\tilde{c}\tilde{f}}(gd)\tilde{h} \xrightarrow{\text{move } \beta} c\tilde{e}\underline{\tilde{d}}f\underline{e\tilde{g}h\tilde{c}\tilde{f}}dg\tilde{h}$$

Step A:

$$c\tilde{e}\underline{\tilde{d}}f\underline{e}\langle\underline{\tilde{g}h}\rangle\underline{\tilde{c}\tilde{f}}d\langle g\tilde{h}\rangle \xrightarrow{\text{move }\delta} \langle c\tilde{e}\rangle\langle\langle\underline{\tilde{d}}f\rangle\rangle\langle\underline{e\tilde{c}}\rangle\langle\langle\underline{\tilde{f}}d\rangle\rangle \xrightarrow{\text{move }\delta} \varnothing$$

Third isle. Steps A, B, C, D: no move is possible.

The procedure stops; the original knot symbol reduces to the equivalent $x\underline{\tilde{y}}a\underline{\tilde{x}}y\underline{\tilde{a}}$.

These two examples have stressed the efficiency and the handiness of our procedure (compare with the amount of work and time required to compute knot polynomials for such diagrams or even to obtain the final result applying the Reidemeister moves!).

It is also worthwhile to notice that, by reconstructing the knot diagram for the reduced knot symbol of the last example, one gets the diagram in Fig. 17 (up to ER1): this is the mirror image of the trefoil knot previously considered, namely it is a simple knot to the right (see Sections 2.2 and 5.1). It is well known that the trefoil knot is the simplest knotted knot and moreover that it is not equivalent to its mirror image [i.e., it is chiral; see Kauffman (1991)]. Note that the two corresponding knot symbols are different and locally irreducible: thus knot symbols seem to be useful tools to detect "knotness" and chirality: of course this would be rigorous if one could prove that local irreducible knot symbols are really irreducible, in other words, that starting from an arbitrary knot symbol, the above procedure yields a unique result up the equivalence rules ER1–5.

We end this section by remarking that (a) it is possible to apply the above procedure by a simple (and fast) algorithm on a computer and (b) the whole scheme could be easily extended to braids and links.

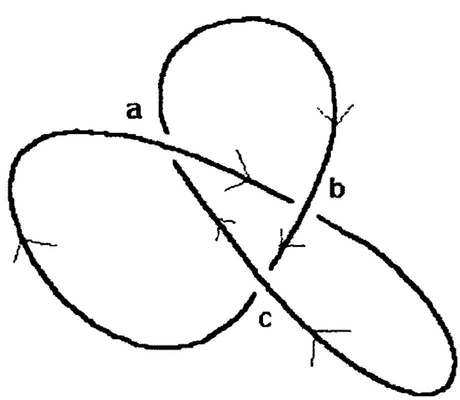

Fig. 17.

5. KNOT SYMBOLS AT WORK

5.1. Simple Knots

First we give the locally irreducible knot symbols for well-known simple knots. Let us start for completeness with the two simplest (knotted) knots even though we already considered them in Sections 2.2 and 4.2.

- The *trefoil knot* or *simple knot to the right* (see Fig. 17): knot symbol $\underline{\tilde{a}}b\underline{\tilde{c}}a\underline{\tilde{b}}c$.
- The *trefoil knot* or *simple knot to the left* (see Fig. 18): knot symbol $\underline{a}\tilde{b}\underline{c}\tilde{a}\underline{b}\tilde{c}$.

Note again that this diagram (Fig. 18) is the mirror image of that in Fig. 17; the mirror in this case is on the right of the first diagram. The two diagrams correspond to different knots [trefoil is chiral (Kauffman, 1991); the corresponding knot symbols are locally irreducible and not equivalent, each one being equivalent to the up or the down conjugate of the other (see considerations in Section 2.1 and at the end of Section 4.2).

- The *figure-eight knot* (see Fig. 19A): knot symbol $\tilde{a}\underline{\tilde{d}}c\underline{a}\tilde{b}\underline{\tilde{c}}d\underline{b}$.

This down alternating knot is known also via the different looking but equivalent diagram in Fig. 19B (Kauffman, 1991); the corresponding knot symbol is $\underline{\tilde{x}}y\underline{\tilde{u}}\tilde{v}\underline{\tilde{y}}x\underline{\tilde{v}}\tilde{u}$, which is equivalent to the previous one through ER1 and ER2. Note how easily we recognize the equivalence of the two diagrams using knot symbols (instead of Reidemeister moves).

Let us now consider the mirror image of the first diagram for the figure-eight knot (see Fig. 20a): the corresponding knot symbol is $a'\underline{d}'\tilde{c}'\underline{\tilde{a}}'b'\underline{c}'d'\underline{b}'$: setting $a' = d$, $b' = c$, $c' = a$, $d' = b$ (ER1) and using ER2, we easily recover the original knot symbol (note, however, that due to the symmetry of the knot, there is also another equivalent setting).

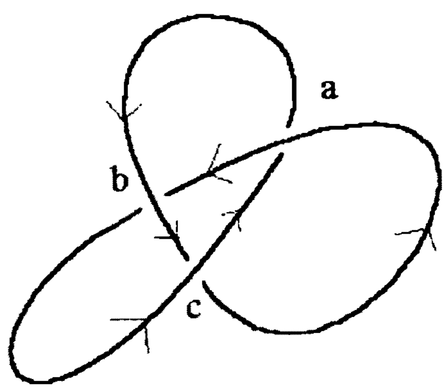

Fig. 18.

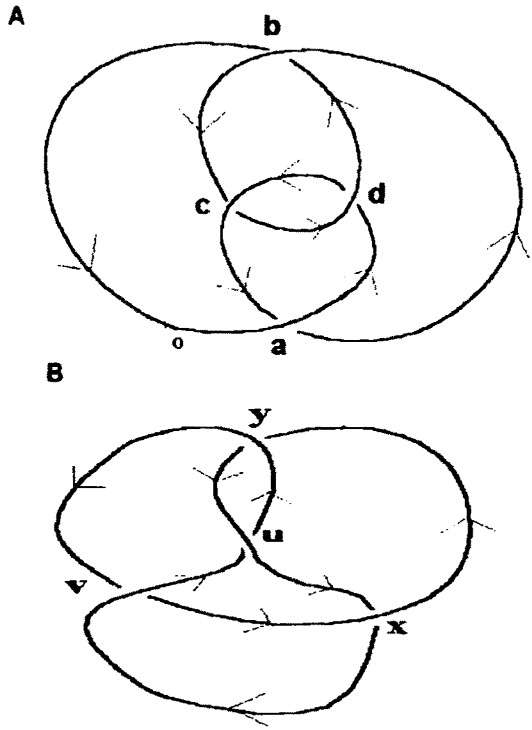

Fig. 19.

Thus we have easily found through knot symbols that the figure-eight knot is achiral, i.e., it can be continuously deformed into its mirror image.

This is indeed well known, but the amount of intuition (and work) that is necessary to go from one diagram to the other using the Reidemeister moves is surprisingly high; here we obtain the result just by looking at the two knot symbols. Moreover, the correspondence between the letters of the two knot symbols (diagrams) helps enough if one tries to deform physically the real, solid knot from one configuration to the other (another not easy task) (see Figs. 20A–20F). The above considerations and those in Sections 2.1 and 4.2 suggest the following criterion to recognize achirality:

AC. If a knot symbol is equivalent to its up and to its down conjugate, then the corresponding knot is achiral.

Remark. Of course it is convenient to test the equivalence by using a locally irreducible expression of the knot symbol. Note also that the possible failure in recognizing the equivalence between two knot symbols (see consid-

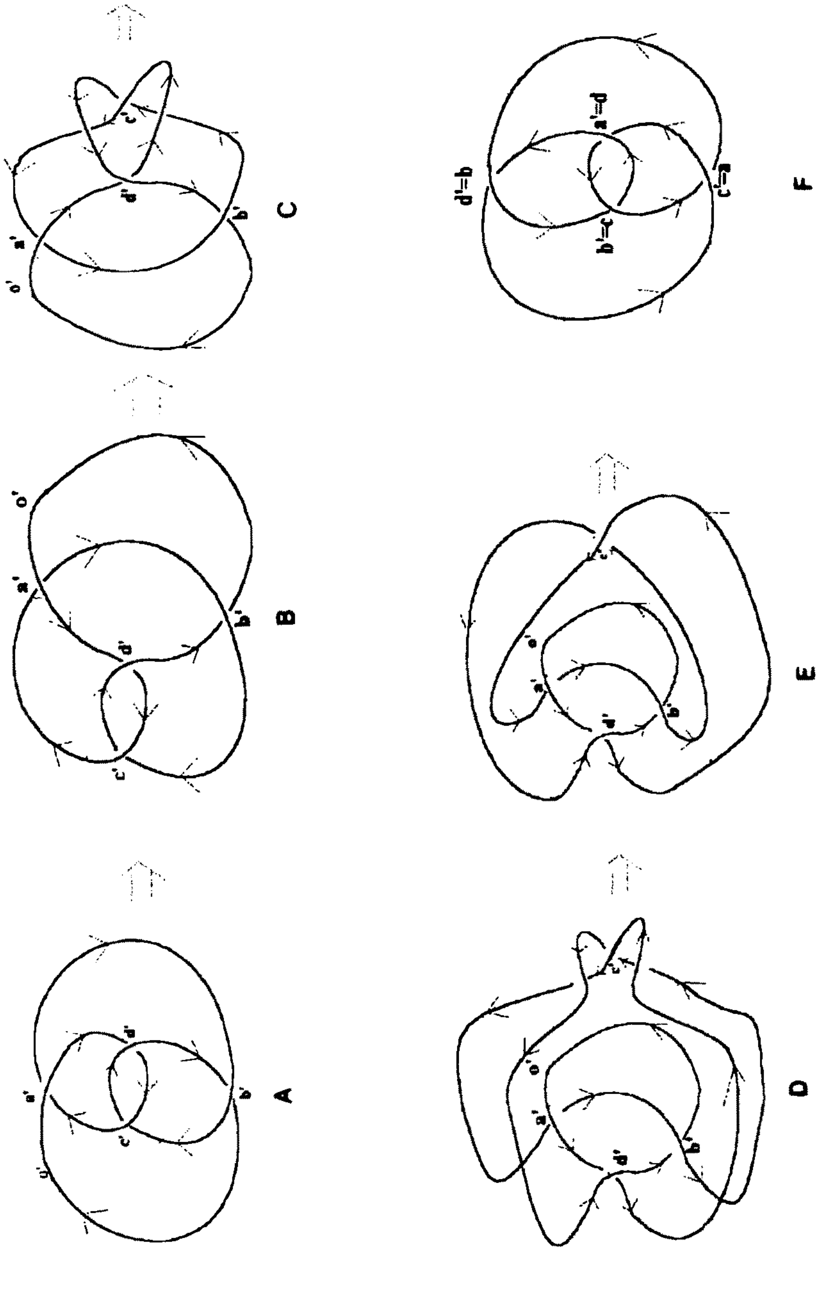

Fig. 20.

erations in Sections 4 and 6) does not allow us to use fully the above criterion: loosely speaking, we can detect achirality, but not chirality.

For the figure-eight knot symbol the above criterion can be easily applied:

- Equivalence with the up conjugate:

$$\underline{a}\tilde{b}\underline{\tilde{c}}d\underline{b}\tilde{a}\underline{\tilde{d}}c \xrightarrow{\text{ER2}} \underline{\tilde{d}}c\underline{a}\tilde{b}\underline{\tilde{c}}d\underline{b}\tilde{a} \xrightarrow{\text{ER1}} \underline{\tilde{a}}b\underline{c}\tilde{d}\underline{\tilde{b}}a\underline{d}\tilde{c}$$

- Equivalence with the down conjugate:

$$\underline{a}\tilde{b}\underline{\tilde{c}}d\underline{b}\tilde{a}\underline{\tilde{d}}c \xrightarrow{\text{ER2}} \underline{\tilde{c}}d\underline{b}\tilde{a}\underline{\tilde{d}}c\underline{a}\tilde{b} \xrightarrow{\text{ER1}} \underline{\tilde{b}}a\underline{c}\tilde{d}\underline{\tilde{a}}b\underline{d}\tilde{c} \xrightarrow{\text{ER3}} a\underline{\tilde{b}}\tilde{c}\underline{d}b\underline{\tilde{a}}\tilde{d}\underline{c}$$

- *The square knot* (see Fig. 21A): knot symbol $\underline{\tilde{a}}b\underline{\tilde{c}}\underline{x}\tilde{y}\underline{z}\tilde{x}\underline{y}\tilde{z}a\underline{\tilde{b}}c$.

This knot symbol contains two proper isles, namely $I_1 = \underline{x}\tilde{y}\underline{z}\tilde{x}\underline{y}\tilde{z}$ and its complementary one: $I_1^c = a\underline{\tilde{b}}c\underline{\tilde{a}}b\underline{\tilde{c}}$ (as is evident also from the diagram in Fig. 21A). Note that one isle is the up or the down conjugate of the other; then it is clear that the knot symbol is equivalent to its partial conjugate (partial conjugation yielding just permutation of the two isles): thus, according to the previous criterion, the square knot is achiral. This also is well known: however, for completeness, let us exhibit (in Fig. 21B) the mirror image of the diagram in Fig. 21A and the corresponding knot symbol:

$$\underline{a}'\tilde{b}'\underline{c}'\underline{\tilde{x}}'y'\underline{\tilde{z}}'x'\underline{\tilde{y}}'z'\tilde{a}'\underline{b}'\tilde{c}'$$

This new knot symbol is evidently the up conjugate of the previous one; moreover, setting a′ = x, b′ = y, c′ = z, x′ = a, y′ = b, z′ = c, it is easily seen that the two symbols are indeed also *completely conjugate* and thus equivalent (ER4). Indeed it is sufficient to make a π rotation in the plane to obtain the new diagram from the previous one (this incidentally suggests a new identification, namely a′ = z, b′ = y, c′ = x, x′ = c, y′ = b, z′ = a, yielding $\underline{z}\tilde{y}\underline{x}\underline{\tilde{c}}b\underline{\tilde{a}}c\underline{\tilde{b}}a\tilde{z}\underline{y}\tilde{x}$, which is equivalent to the original one due to ER2, ER3). It is worthwhile to remark that a physical twist (π rotation in space) of an isle (say I_1) in the real knot gives the diagram in Fig. 21C: the corresponding knot symbol $\underline{\tilde{a}}b\underline{\tilde{c}}\underline{\tilde{x}}y\underline{\tilde{z}}x\underline{\tilde{y}}za\underline{\tilde{b}}c$ is immediately seen to be equivalent to the original one through the move $\alpha 2$.

The granny knot (see Fig. 22): knot symbol: $\underline{\tilde{a}}b\underline{\tilde{c}}x\underline{\tilde{y}}z\underline{\tilde{x}}y\underline{\tilde{z}}a\underline{\tilde{b}}c$.

Again we have two locally irreducible isles; note also that the knot symbol for the granny knot can be obtained from that of the square knot by a down partial conjugation of its isle I_1: indeed the above diagram is obtained from that of the square knot by substitution of its right-hand side with its mirror image (from below). It is well known that this knot is chiral: it is satisfying to notice that indeed the knot symbol for the granny knot does not

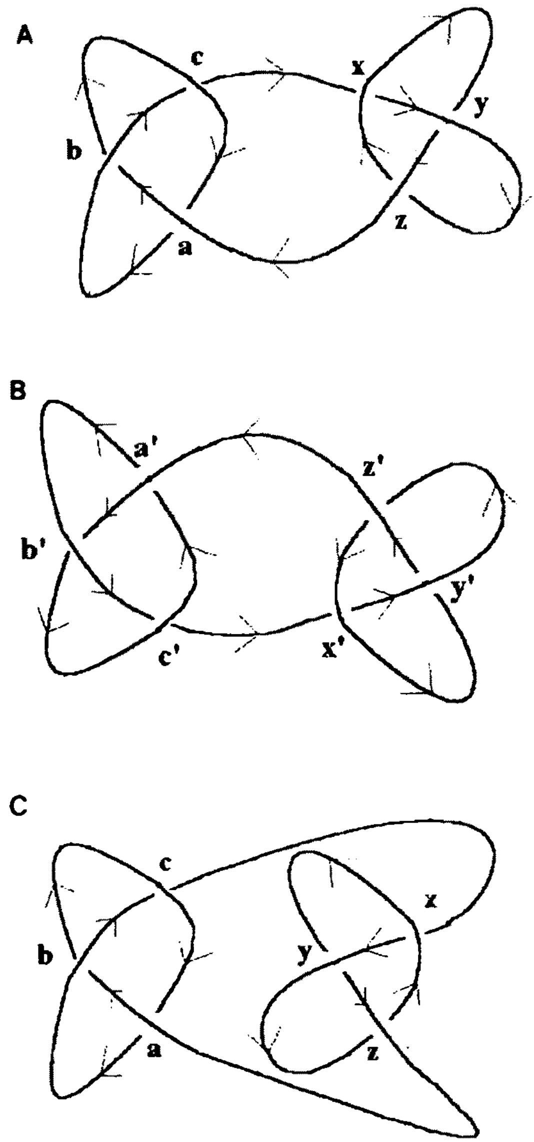

Fig. 21.

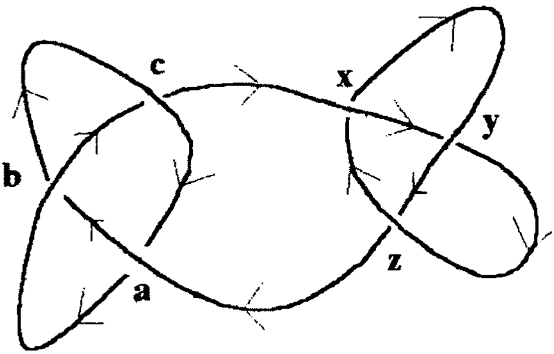

Fig. 22.

satisfy the achirality criterion AC [the knot symbol being up and down alternating; see Remark (iii) of Section 4.1].

5.2. *Hard* Knots: Testing the Procedure

We want now to test the efficiency of our procedure with some interesting knots. Let us start with the classical example of a knot for which there is no possibility to reduce the number of crossings through the Reidemeister moves: nevertheless the knot is unknotted (indeed, using first in a proper way the Reidemeister moves to add crossings, it is possible to reduce the knot to the unknotted). The diagram of such knot is shown in Fig. 23 (Lickorish, 1988); its knot symbol is $\underline{a}\tilde{\bar{b}}\underline{c}\underline{\tilde{\bar{d}}}e\underline{\tilde{\bar{f}}}g\tilde{\bar{a}}\underline{b}\underline{\tilde{\bar{e}}}f\underline{\tilde{\bar{g}}}d\tilde{\bar{c}}$. Following our procedure, we have

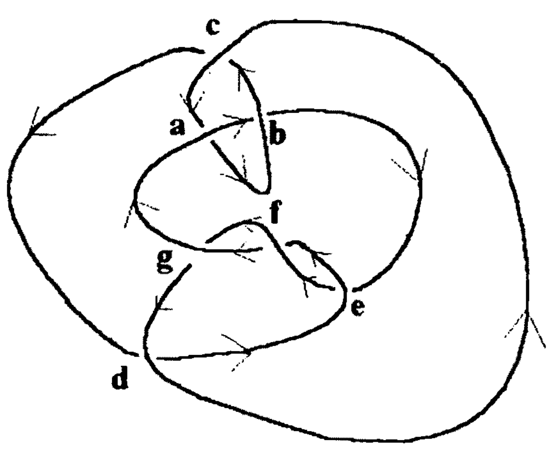

Fig. 23.

$$\underline{a}\tilde{b}\langle\underline{c\tilde{d}}\rangle e\underline{\tilde{f}}g\tilde{a}\underline{b\tilde{e}}f\underline{\tilde{g}}\langle d\tilde{c}\rangle \xrightarrow{\text{move }\delta} \underline{a}\langle\tilde{b}e\rangle\underline{\tilde{f}}g\tilde{a}\langle\underline{b\tilde{e}}\rangle f\underline{\tilde{g}}$$

$$\xrightarrow{\text{move }\delta} \underline{a\tilde{f}}g\tilde{a}f\underline{\tilde{g}} \xrightarrow{\text{ER2}} \langle\underline{\tilde{g}a}\rangle\underline{\tilde{f}}\langle g\tilde{a}\rangle f$$

$$\xrightarrow{\text{move }\delta} \{\underline{\tilde{f}}f\} \xrightarrow{\text{move }\gamma} \varnothing$$

The secret of this success is that our move δ is not just the Reidemeister move II, but includes also a convenient combination of the other Reidemeister moves as well.

Let us now apply our scheme to a knot and its *mutant* (see Figs. 24A and 24B) (Lickorish, 1988). These two knots have the same (trivial) Alexander Polynomial and also the same HOMFLY Polynomial.

The mutant is obtained by a π rotation in the plane of the dotted ball. Let us construct now the two knot symbols S for the knot in Fig. 24A and SM for the mutant in Fig. 24B:

$$S \equiv \underline{\tilde{a}}\tilde{b}\underline{cx}\tilde{y}\underline{z}u\underline{\tilde{v}}\tilde{x}\underline{y}\tilde{z}\tilde{d}\underline{b}e\underline{\tilde{f}}\tilde{c}\underline{d}a\underline{\tilde{e}}fv\underline{\tilde{u}}$$

$$SM \equiv \underline{\tilde{a}}\tilde{b}\underline{c\tilde{u}}v\tilde{d}\underline{b}e\underline{\tilde{f}}\tilde{c}\underline{d}a\underline{\tilde{e}}f\tilde{z}\underline{y}\tilde{x}\underline{\tilde{v}}u\underline{z}\tilde{y}\underline{x}$$

The two knot symbols are locally irreducible: indeed no isle, noose, snare, or chain can be found; on the other hand, the two symbols are not equivalent via the equivalence rules ER1–4.

Let us give another example of two different knots which have the same HOMFLY Polynomial but different locally irreducible knot symbols: their diagrams are given in Figs. 25A and 25B. The two diagrams have a different number of crossings, and consequently the corresponding knot symbols

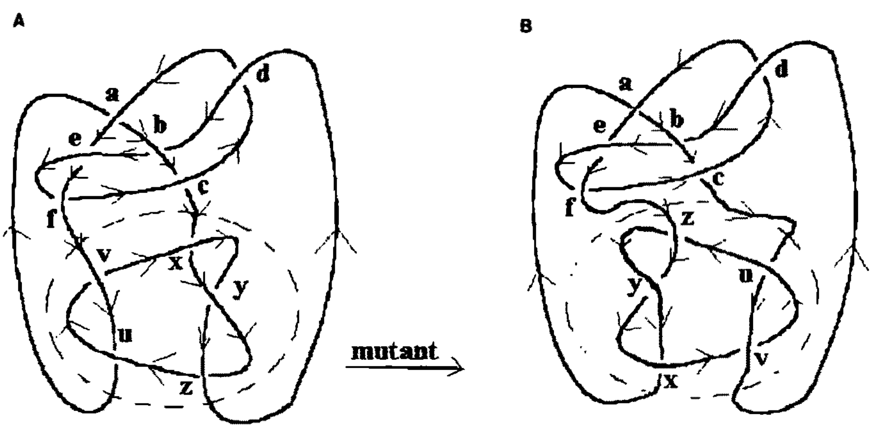

Fig. 24.

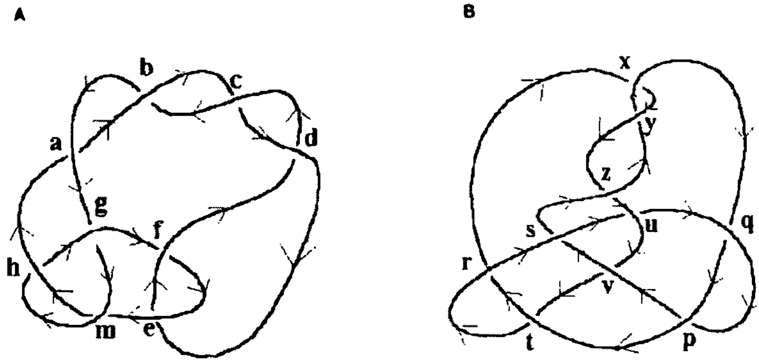

Fig. 25.

(Sa and Sb, respectively) have a different number of letters: nevertheless it could happen that they reduce to the same knot symbol following our procedure. Let us examine thus:

$$Sa = \underline{\tilde{a}}b\underline{\tilde{c}}d\underline{e}\tilde{f}\underline{\tilde{d}}c\underline{\tilde{b}}a\underline{g}\tilde{m}\underline{h}\tilde{g}\underline{f}\tilde{e}\underline{m}\tilde{h}$$

$$Sb = \underline{x}\tilde{y}\underline{z}u\underline{\tilde{v}}\underline{\tilde{t}}rs\underline{\tilde{u}}\tilde{q}\underline{p}v\underline{\tilde{s}}\tilde{z}\underline{y}\tilde{x}\underline{q}\tilde{p}t\underline{\tilde{r}}$$

- Sa is down alternating with no isle or noose, thus it is locally irreducible.
- Sb has no isle or noose or snare, but it exhibits a chain, so we can apply the move β:

$$\underline{x}\tilde{y}\underline{z}u(\underline{\tilde{v}}\underline{\tilde{t}})rs\underline{\tilde{u}}\tilde{q}(\underline{p}v)\underline{\tilde{s}}\tilde{z}\underline{y}\tilde{x}\underline{q}(\tilde{p}t)\underline{\tilde{r}}$$

$$\xrightarrow{\text{move } \beta} \underline{x}\tilde{y}\underline{z}u\underline{\tilde{t}}\underline{\tilde{v}}rs\underline{\tilde{u}}\tilde{q}v\underline{p}\underline{\tilde{s}}\tilde{z}\underline{y}\tilde{x}\underline{q}t\tilde{p}\underline{\tilde{r}}$$

Now a different chain is exhibited:

$$\underline{x}\tilde{y}\underline{z}u\underline{\tilde{t}}\underline{\tilde{v}}(rs)\underline{\tilde{u}}\tilde{q}v(\underline{p}\underline{\tilde{s}})\tilde{z}\underline{y}\tilde{x}\underline{q}t(\tilde{p}\underline{\tilde{r}})$$

$$\xrightarrow{\text{move } \beta} \underline{x}\tilde{y}\underline{z}u\underline{\tilde{t}}\underline{\tilde{v}}sr\underline{\tilde{u}}\tilde{q}v\underline{\tilde{s}}\underline{p}\tilde{z}\underline{y}\tilde{x}\underline{q}t\underline{\tilde{r}}\tilde{p}$$

Again a new chain is generated, so applying the move β, we get

$$\underline{x}\tilde{y}\underline{z}u(\underline{\tilde{t}}\underline{\tilde{v}})sr\underline{\tilde{u}}(\tilde{q}v)\underline{\tilde{s}}\underline{p}\tilde{z}\underline{y}\tilde{x}(\underline{q}t)\underline{\tilde{r}}\tilde{p}$$

$$\xrightarrow{\text{move } \beta} \underline{x}\tilde{y}\underline{z}u\underline{\tilde{v}}\underline{\tilde{t}}(sr)\underline{\tilde{u}}v(\tilde{q}\underline{\tilde{s}})\underline{p}\tilde{z}\underline{y}\tilde{x}t(\underline{q}\underline{\tilde{r}})\tilde{p}$$

$$\xrightarrow{\text{move } \beta} \underline{x}\tilde{y}\underline{z}u\underline{\tilde{v}}\underline{\tilde{t}}rs\underline{\tilde{u}}v\underline{\tilde{s}}\tilde{q}\underline{p}\tilde{z}\underline{y}\tilde{x}t\underline{\tilde{r}}\underline{q}\tilde{p}$$

No other move is possible without going back, thus also this knot symbol is locally irreducible and moreover it is manifestly not equivalent to the previous one through the equivalence rules ER1–4: thus again different knots not distinguished by the HOMFLY Polynomials have different locally irreducible knot symbols.

6. CONCLUSIONS

Let us briefly summarize: the knot symbols are a convenient tool to handle the reduction of a knot; give a knot diagram, the starting knot symbol is easily constructed through rules R1–4; manipulating this knot symbol via the introduced fast and easy procedure, one gets a final reduced equivalent knot symbol that allows one to reconstruct a simplified diagram equivalent to the original one but with a smaller or at least not greater number of crossings. This diagram is hopefully minimal, but even if this were not true, the previous examples should prove the usefulness and manageability of our scheme in classifying and simplifying knot diagrams.

Even if one is working with knot polynomials, one should take advantage of the possibility of starting with a simplified diagram (a great advantage indeed, considering the lengthy procedures needed for the construction of knot polynomials).

It is worthwhile also to remark that our scheme is in a sense dual with respect to using invariants (polynomials): indeed this duality is well expressed by the following scheme, where K denotes a knot, P(K) an invariant (polynomial) for the knot, and S(K) the corresponding knot symbol:

$$P(K_1) \neq P(K_2) \Rightarrow K_1 \neq K_2$$

$$P(K_1) = P(K_2) \not\Rightarrow K_1 = K_2$$

$$S(K_1) \neq S(K_2) \not\Rightarrow K_1 \neq K_2$$

$$S(K_1) = S(K_2) \Rightarrow K_1 = K_2$$

In other words, different knots can have the same invariant, and equivalent knot symbols correspond to the same knot [see Remark (i) of Section 4.1]. Also, this duality should indicate that the two approaches are not alternatives, but could be profitably used together. Finally we mention again that our procedure to simplify knot symbols can easily be implemented on a computer and that the whole scheme can be extended to more complex objects such as braids and links.

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